Section 2.2 HW problems

1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $\mathsf{T}, \mathsf{U}: \mathsf{V} \to \mathsf{W}$ are linear transformations.

(a) For any scalar a, $a\mathsf{T} + \mathsf{U}$ is a linear transformation from V to W. TRUE

(b) $[\mathsf{T}]_\beta^\gamma = [\mathsf{U}]_\beta^\gamma$ implies that $\mathsf{T} = \mathsf{U}$. TRUE

(c) If $m = \dim(\mathsf{V})$ and $n = \dim(\mathsf{W})$, then $[\mathsf{T}]_\beta^\gamma$ is an $m \times n$ matrix.

(d) $[\mathsf{T}+\mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$. TRUE

(e) $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is a vector space. TRUE

(f) $\mathcal{L}(\mathsf{V}, \mathsf{W}) = \mathcal{L}(\mathsf{W}, \mathsf{V})$. False

→ $n \times m$ matrix
- number of rows = dim W = n
- number of columns = dim V = m

$[T]_\beta^\gamma = n \times m$ matrix.

* If $T: R^2 \to R^3$, then $m = 2, n = 3$, $3 \times 2 = [T]$

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $\mathsf{T}: \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.

(a) $\mathsf{T}: \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 4 & 0 \end{pmatrix}$$

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

#5-(b)

$T: P_2(R) \to M_{2\times2}(R)$

$\{1, x, x^2\}$

$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$

① Using β

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

② Using α

$0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Define

$$\mathsf{T}: \mathsf{P}_2(R) \to \mathsf{M}_{2\times2}(R) \quad \text{by} \quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$.

(c) Define $\mathsf{T}: \mathsf{M}_{2\times2}(F) \to F$ by $\mathsf{T}(A) = \mathrm{tr}(A)$. Compute $[\mathsf{T}]_\alpha^\gamma$.

the sum of its diagonal elements

① using α $\quad (1, 0, 0, 1)$

② Using γ

$[T]_\alpha^\gamma = (1, 0, 0, 1)$

10. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6 (p. 72), there exists a linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[\mathsf{T}]_\beta$.

$T(V_1) = V_1 + V_0 = V_1$

$T(V_2) = V_2 + V_1$

$T(V_3) = V_3 + V_2$

$\vdots$

$T(V_n) = V_n + V_{n-1}$

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & & & \vdots \\ & & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & & & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

16. Let V and W be vector spaces such that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix.

$V \Rightarrow \beta = \{e_1, e_2 \ldots, e_n\}$ since $\dim(V) = \dim(W)$

$\gamma = \{e_1, e_2 \ldots e_n\}$

2.7
pt.2

September 26, FRI

Kronecker notation

$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & i \neq j \end{cases}$

$(I_n)_{ij} = \delta_{ij}$ n×n identity matrix

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Properties: ① $A(B+C) = AB + AC$ ← distribution law for matrix

② $a(AB) = (aA)\cdot B = A\cdot(aB)$ ← Also distribution law for matrix

number

③ $IA = A$ $AI = A$ Since I is identity matrix

dimension in basis β

④ $I_V : V \to V$ identity transformation

$I_V(\vec{x}) = \vec{x}$

$[I_V]_\beta = I_n$

regarding ① matrix multiplication ← assume the size is matched

Note: $A^2 = A\cdot A$

$A^3 = A\cdot A^2$

$A^0 = I_n$

Cancellation law $AB = AC \to B = C$ but doesn't hold for matrix multiplication ∵ Not valid for Matrix multiplication

② No cancellation law for matrix multiplication.

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\nRightarrow A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Theorem. $A_{m\times n}$, $B_{n\times p}$

Let U_j = jth column of AB

V_j = " " " B

Then (1) $U_j = A\cdot V_j$

(2) $V_j = Be_j$ where $e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ → jth

the prof says he will remind us about this

pf. (1) $U_j = \begin{pmatrix} (AB)_{1j} \\ (AB)_{2j} \\ \vdots \\ (AB)_{mj} \end{pmatrix}$ ← jth column of AB

first row · vj

second row · vj

$= \begin{pmatrix} \sum_{k=1}^{n} A_{1k} B_{kj} \\ \sum_{k=1}^{n} A_{2k} B_{kj} \\ \vdots \\ \sum_{k=1}^{n} A_{mk} B_{kj} \end{pmatrix} = A\cdot V_j$

(2) apply (1) to $BI = B$

Sample Basis β Basis γ

Theorem. $T: V \to W$ $\vec{u} \in V$, $T(\vec{u}) \in W$

Then $[T(\vec{u})]_\gamma = [T]_\beta^\gamma \cdot [\vec{u}]_\beta$

Note: coordinate vector of output = matrix representation × coordinate vector of original output

Ex. $T: P_3 \to P_2$ by $T(f(x)) = f'(x)$

$\beta = \{1, x, x^2, x^3\}$ $\gamma = \{1, x, x^2\}$

Let $p(x) = 2 - 4x + x^2 + 3x^3 \in P_3$

Verify $[T(p(x))]_\gamma = [T]_\beta^\gamma \cdot [p(x)]_\beta$

pf. $[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$[p(x)]_\beta = \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix}$

|| coordinate of β

right side

$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$

same

left side

$T(p(x)) = -4 + 2x + 9x^2$ $\{1, x, x^2\}$

$[T(p(x))]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$

↓

coordinate of γ

Def. Given $A_{m\times n}$

Define $L_A : R^n \to R^m$

by $L_A(\vec{x}) = \underbrace{A}_{m\times n}\underbrace{\vec{x}}_{n\times 1}$ where $\vec{x} \in R^n$

L_A is called left-multiplication transformation

Theorem: (1) $[L_A]_\beta^\gamma = A_{m\times n}$ $\longrightarrow$ check the reminder $A\cdot e_1 =$ first column

(2) $L_A = L_B$ iff $A = B$

(3) $L_{A+B} = L_A + L_B$

(4) $L_{AD} = L_A \circ L_D$

$A_{m\times n}$

$B_{m\times n}$

$D_{n\times p}$

Theorem:

$(AB)C = A(BC)$ A,B,C matrices

Association

pf. $L_{A(BC)} = L_A \circ L_{BC} = L_A \circ (L_B \circ L_C)$

$= (L_A \circ L_B) \circ L_C$

$= L_{AB} \circ L_C$

$= L_{(AB)C}$

4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.14 to compute the following vectors:

(a) $[\mathsf{T}(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

(b) $[\mathsf{T}(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

(b) Define

$$\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{M}_{2\times 2}(R) \quad \text{by} \quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

$\boxed{1}$ first, we need to define $[T]_\beta^\alpha$

$T: P_2(R) \to M_{2\times2}(R)$

$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$

$\beta = \{1, x, x^2\}$

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$T(1) = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x^2) = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{4\times3}$

$[f(x)]_\beta = \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}_{3\times1}$

$\boxed{2}$ for the (b) $[T(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$

$[T(f(x))]_\alpha = [T]_\beta^\alpha \cdot [f(x)]_\beta$

$= \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ ☒

$\boxed{3}$ for the (a) $[T(A)]_\alpha$ where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$

Let's compute $[A]_\alpha$ first.

$[A]_\alpha = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix}$

Why? $[T]_\alpha^\alpha$ to $[T]_\alpha$

Since in Exercise 5, $T: M_{2\times2}(F) \to M_{2\times2}(F)$ is $T(A) = A^t$

By theorem, $[T(A)]_\alpha = [T]_\alpha^\alpha [A]_\alpha$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ "11=11"

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ "12=21"

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ 21=12

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ 22=22

$\to [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}$

$[T(A)]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}_{4\times1}$

9. Find linear transformations $\mathsf{U}, \mathsf{T}\colon \mathsf{F}^2 \to \mathsf{F}^2$ such that $\mathsf{UT} = \mathsf{T}_0$ (the zero transformation) but $\mathsf{TU} \neq \mathsf{T}_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

sol)

Let $U(x,y) = (y,0)$
$T(x,y) = (x,0)$

U takes (x,y) and keeps second component...
Ex. $U(3,5) = (5,0)$

T takes (x,y) and keeps only first component
Ex $T(3,5) = (3,0)$

Then $UT(x,y) = U(x,0) = (0,0)$
so $UT = T_0$ as required

similarly, $TU(x,y) = T(y,0) = (y,0)$
if $y \neq 0$ then $(y,0)$ is not the zero vector

Therefore $TU \neq T_0$, as required

Let β be the standard basis
Then let $A = [U]_\beta$
$B = [T]_\beta$

That is
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ it follows that

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O$, as required.

11. Let V be a vector space, and let $T: V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

12. Let V, W, and Z be vector spaces, and let $T: V \to W$ and $U: W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^t)$.

#11.

$R(T)$ = range (all outputs of T)

$N(T)$ = null space (all inputs that T maps to zero)

[1] Assume $R(T) \subseteq N(T)$

Then for all $v \in V$, $Tv \in R(T) \subseteq N(T)$

so $T^2v = T(Tv) = 0$

As v is arbitrary $T^2 = 0$

[2] suppose $T^2 = 0$. Then for all $v \in R(T)$ there exists $w \in V$ such that $v = Tw$, $Tv = T^2w = 0$

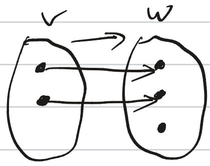
$v \in N(T)$. As v is arbitrary, we have $R(T) \subseteq N(T)$

Therefore, $T^2 = 0$ iff $R(T) \subseteq N(T)$

#12 $T: V \to W$

$U: W \to Z$

(a) prove that if UT is one-to-one, then T is one-to-one. → at most one solution $n \le m$



Must U also be one-to-one?

Let $v \in N(T)$. Then $Tv = 0$, so $UT(v) = 0$.

As UT is one-to-one, $v = 0$ so T is also one-to-one.

UT = Id (identity map)

refer to 2.1.21

(b) Let $z \in Z$. As UT is onto, there exists $v \in V$ such that

$z = UT(v) = U(T(v)) \in R(U)$

As z is arbitrary, U is onto.

(c) U, T one-to-one & onto

Let $z \in Z$. Then there exists $w \in W$ such that $z = Uw$. As T is onto, there exists $v \in V$ such that $Tv = w$. So $z = Uw = U(Tv) = (UT)(v) \in R(UT)$. As z is arbitrary, UT is onto

Let $v \in N(UT)$. Then $0 = UT(v) = U(Tv)$. so $Tv \in N(U) = \{0\}$.

Hence $Tv = 0$, $v \in N(T)$, $v = 0$. As v is arbitrary, UT is one-to-one

Hence UT is one-to-one & onto

#13

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

$$\mathrm{tr}(A^t) = \sum_{i=1}^{n} (A^t)_{ii} = \sum_{i=1}^{n} (A)_{ii} = \mathrm{tr}(A)$$

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} A_{ij} \sum_{j=1}^{n} B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji} A_{ij} = \sum_{j=1}^{n} (BA)_{jj}$$

$$= \mathrm{tr}(BA)$$

September 29, Monday

§2.4 Invertibility & Isomorphism

Ex. ① $f(x) = x^2$

$g(x) = \sqrt{x}$ $f(g(x)) = x$

② $f(x) = 2x$

$g(x) = \frac{1}{2}x$

Similar to invertible? inverse? function

Def $T: V \to W$

$U: W \to V$

U is called the inverse of T if $TU = I_W$ and $UT = I_V$

Ex. $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ by $T(a+bx) = (a, a+b)$

find $T^{-1}: \mathbb{R}^2 \to P_1(\mathbb{R})$

find $(c,d) \to ?$

$f(2x, y) = 4x^2 + 2y$
$f(x,y) = x^2 + 2y$

Pf. $UT(a+bx) = a+bx$

$= U(a, a+b)$

$U(a, a+b) = a + bx$

$a = c$, $a+b = d$ $\Rightarrow a = c$, $b = d-c$

final result.

$\Rightarrow U(c,d) = c + (d-c)x$

properties: ① $(TU)^{-1} = U^{-1} \circ T^{-1}$

② $(T^{-1})^{-1} = T$

useful one ③ T is invertible $\Leftrightarrow$ T is one to one and onto

Proof ①: $(U^{-1} \circ T^{-1}) \circ (T \circ U) = U^{-1} \circ (T^{-1} \circ T) \circ U$

$= U^{-1} \circ I \circ U$

$= U^{-1} \circ U$

$= I$

* $T: V \to W$
$U: W \to V$
$U^{-1}: V \to W$
$T^{-1}: W \to V$

$(T \circ U) \circ (U^{-1} \circ T^{-1}) = T \circ (U \circ U^{-1}) \circ T^{-1}$

$= T \circ I \circ T^{-1}$

$= T \circ T^{-1}$

$= I$

② ✓

③ $\Rightarrow$ suppose $T: V \to W$

$\exists U: W \to V$ st $UT = I$ $TU = I$

inverse transformation

Q. Why T is one to one?

If $T(x) = T(y)$, then $U(T(x)) = U(T(y))$
$U(T(x)) = x$, $U(T(y)) = y$ $\Rightarrow x = y$

Q. why T is onto?

$\forall \vec{w} \in W$, want to show $\exists \vec{v} \in V$

st $T(\vec{v}) = \vec{w}$

$\vec{w} = TU(\vec{w}) = T(U(\vec{w}))$

let $\vec{v} = U(\vec{w})$, then $T(\vec{v}) = \vec{w}$

Def $A_{n\times n}$ is invertible if $\exists B_{n\times n}$ such that $AB = BA = I_n$

denote $B = A^{-1}$

Ex $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ $A^{-1} = ?$

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = A^{-1}$

$5a + 7c = 1$ $2a + 3c = 0$
$5b + 7d = 0$ $2b + 3d = 1$

solve by matrix multiplication

$$\boxed{\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} = \frac{1}{xw - yz}\begin{pmatrix} w & -y \\ -z & x \end{pmatrix}}$$

* remember this formula

$xw - yz$: determinant

Note: the inverse A^{-1} is unique

finite-dim vector spaces

Note: T is invertible $V \to W$

Then $\dim(V) = \dim(W)$

pf: $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

$\text{nullity}(T) = \dim(N(T)) = 0$ bc T is one to one

$\text{rank}(T) = \dim(R(T)) = \dim(W)$ bc T is onto

Here $R(T) = W$

$0 + \dim(W) = \dim(V)$

$\therefore$ two vector spaces have same dimension

Theorem. $T: \underset{\beta}{V} \to \underset{\gamma \text{ basis}}{W}$

T is invertible $\Longleftrightarrow$ $[T]_\beta^\gamma$ is invertible

furthermore, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$

$I_n = [I_V]_\beta = [T^{-1} \circ T]_\beta$

$= \underbrace{[T^{-1}]_\gamma^\beta \cdot [T]_\beta^\gamma}_{}$

one is inverse of the other

2.5 change of Coordinate matrix

$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ Coordinate vector in relative to basis β

Q. what happens if we change the basis?

Theorem. Let β' and β be two orderded bases for vec space V

$I_V : V \to V$ identity transformation
$\beta' \quad \beta$

Let $Q = [I_V]_{\beta'}^{\beta}$

Then (1) Q is invertible

(2) $\forall \vec{v} \in V$, $[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}$

Pf. (1) I_V is invertible $\Rightarrow$ Q is invertible

(2) $[\vec{v}]_\beta = [I_V(\vec{v})]_\beta = [I_V]_{\beta'}^{\beta} \cdot [\vec{v}]_{\beta'}$

$= Q \cdot [\vec{v}]_{\beta'}$

Note: ① Q is called the change of Coordinates matrix

It changes β'- coordinates to β- coordinates

② If Q changes β' to β than Q^{-1}... β to β'.

Ex. R^2 $\beta' = \{(2,4), (3,1)\}$, $\beta = \{(1,1), (1,-1)\}$

find $Q = [I_V]_{\beta'}^{\beta}$

sol) $I_V\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\end{pmatrix}$

$= 3\begin{pmatrix}1\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\-1\end{pmatrix}$

bc identity!

$I_V\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\-1\end{pmatrix}$

$\Rightarrow Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

Theorem.

$T: V \to V$ T: linear operator
$\beta' \quad \beta$

Then $[T]_{\beta'} = Q^{-1} \cdot [T]_\beta \cdot Q$ where $Q = [I_V]_{\beta'}^{\beta}$

$Q[T]_{\beta'} = [T]_\beta Q$

pf. $I_V : V \to V$
$\beta' \quad \beta$

Composition

check the 2.3!! it's main idea!!

Left $= Q[T]_{\beta'} = [I]_{\beta'}^{\beta} \cdot [T]_{\beta'}^{\beta'} = [I \circ T]_{\beta'}^{\beta}$

$= [T \cdot I]_{\beta'}^{\beta} = [T]_\beta^\beta \cdot [I]_{\beta'}^{\beta} = [T]_\beta \cdot Q =$ right

Def. $A, B \in M_{n\times n}$

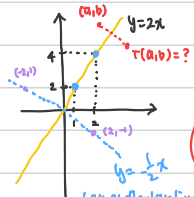
B is similar to A if $\exists Q$ (invertible) st $B = Q^{-1}AQ$

Note: $[T]_\beta$ is similar to $[T]_{\beta'}$

Ex. Reflection problem (He said it's interesting)

find the reflection T about line $y = 2x$

1



$T\begin{pmatrix}a\\b\end{pmatrix} = ?$

Let $\beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ standard basis

perpendicular line $= [T]_\beta \cdot \begin{pmatrix}a\\b\end{pmatrix}$

$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ $T\begin{pmatrix}2\\4\end{pmatrix}$? They are dependent

so we choose different points

so $T\begin{pmatrix}-2\\1\end{pmatrix}$ from $y = -\frac{1}{2}x$ which is a perpendicular line to $y=2x$

$T\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

2

* tricky part so chose two points one from line one from perpendicular line. + refer to HW to practice

Let $\beta' = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix} \right\}$ → not parallel. be a basis of R^2

$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

why?

$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$= 1\begin{pmatrix}1\\2\end{pmatrix}$ to $\begin{pmatrix}-2\\1\end{pmatrix}$

$T\begin{pmatrix}-2\\1\end{pmatrix} = a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}-2\\1\end{pmatrix}$

$a - 2b = 2$
$2a + b = -1$

$a + 2 = 2$
$a = 0$

$2a - 4b = 4$
$-)\ 2a + b = -1$
$-5b = 5$
$b = -1$

3

$Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

target

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$[T]_\beta = Q[T]_{\beta'}Q^{-1}$

$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

so $T\begin{pmatrix}a\\b\end{pmatrix} = [T]_\beta \cdot \begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} -3a + 4b \\ 4a + 3b \end{pmatrix}$

2.5 textbook.

2.5 The change of coordinate matrix

" coordinates relative to a basis " mean?

- A vector in R^2 can be written .. depending on the basis we use.

ex).

standard basis $e_1 = (1,0)$ $e_2 = (0,1)$

then the vector $(3,1)$ has coordinate $(3,1)$

But if we use new basis, say $\beta' = \{(2,4),(3,1)\}$, then the vector $(3,1)$ have different coordinates.

* The change of coordinate matrix Q.

If $[v]_{\beta'}$ are the coordinates of a vector v in basis β', then

$[v]_\beta = Q[v]_{\beta'}$.

\+ when we want to compute the matrix of a linear operator T in a new basis, we use:

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

Example 1.

$\beta = \{(1,1),(1,-1)\}$

$\beta' = \{(2,4),(3,1)\}$

$\boxed{1}$ express each vector in β' using β

so we find constants c_1, c_2 .. such that

① $(2,4) = c_1(1,1) + c_2(1,-1)$
$(c_1+c_2,\ c_1-c_2)$

$c_1 + \cancel{c_2} = 2$
$c_1 - \cancel{c_2} = 4$
$2c_1 = 6$
$c_1 = 3$ $c_2 = -1$

So $(2,4) = 3(1,1) - 1(1,-1)$

② $(3,1) = c_1(1,1) + c_2(1,-1)$
$= (c_1+c_2,\ c_1-c_2)$

$c_1 + \cancel{c_2} = 3$
$c_1 - \cancel{c_2} = 1$
$2c_1 = 4$
$c_1 = 2$ $c_2 = 1$

So $(3,1) = 2(1,1) + 1(1,-1)$

$\boxed{2}$ Build the matrix Q

$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ ⇒ That's the change of coordinate matrix takes coordinates from the β'-system to the β-system.

$\boxed{3}$ How to use Q.

If you have a vector v, and you know its coordinate in the β'-basis, say

$[v]_{\beta'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $[v]_\beta = Q[v]_{\beta'} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

So that means, $(2,4) = 3(1,1) - 1(1,-1)$

exactly as we found before.

$\boxed{4}$ Summary: $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

Example 2

<set up>

- Bases from Ex1.

$\beta = \{(1,1),(1,-1)\}$ $\beta' = \{(2,4),(3,1)\}$

$T(a,b) = (3a-b,\ a+3b)$

$\boxed{1}$ Compute $[T]_\beta$

- $v_1 = (1,1)$

$T(1,1) = (2,4) = 3(1,1) - 1(1,-1)$

$\Rightarrow [T(1,1)]_\beta = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

- $v = (1,-1)$

$T(1,-1) = (4,-2) = 1(1,1) + 3(1,-1)$

$\Rightarrow [T(1,-1)]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

$\boxed{2}$ change of coordinates matrix Q. (from $\beta' \to \beta$)

$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ $Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

$\boxed{3}$ Convert the matrix of T to the β' basis

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$[T]_\beta Q = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ -6 & 1 \end{pmatrix}$

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$= \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 8 & 7 \\ -6 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$

Oct 16th 2025 Review for exam 2 (chapter 2)

§2.1 linear trans

$T(c\vec{x}+\vec{y}) = c\cdot T(\vec{x}) + T(\vec{y})$

$N(T)$, $R(T)$

Dim Thm. $\dim(V) = \dim(N(T)) + \dim(R(T))$

T is 1-1 $\Leftrightarrow N(T) = \{\vec{0}\}$

if $\dim(V) = \dim(W)$, 1-1 $\Leftrightarrow$ onto

§2.2 $T: V \to W$ (V: β, n; W: γ, m)

$[T]_\beta^\gamma = [T(\beta)]_\gamma$

§2.3 $U \circ T$

suppose $T: V \to W$ (V: α; W: β) $U: W \to Z$ (W: β; Z: γ)

$[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$ cancel

$[T(\vec{v})]_\beta = [T]_\alpha^\beta [\vec{v}]_\alpha$

$(AB)_{ij} = \sum_k A_{ik} B_{kj}$

$A: m\times n$ $L_A: R^n \to R^m$

$\vec{x} \to A\vec{x}$

§2.4 $T: V \to W$ (V: β; W: γ)

* invertibility = 1-1 and onto

If T is invertible, then

$\dim(V) = \dim(W)$

$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$

β, n γ, m

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ R^n \arrow[r, "L_A"] & R^m \end{tikzcd}$$

$A_{m\times n} = [T]_\beta^\gamma$

$L_A \circ \phi_\beta(\vec{v}) = \phi_\gamma \circ T(\vec{v})$ $\forall \vec{v} \in V$

§2.5 $I_V: V \to W$ (V: β'; W: β)

Identity

$[I_V]_{\beta'}^{\beta} = Q$

$T: V \to V$

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$Q[T]_{\beta'} = QQ^{-1}[T]_\beta Q$

$\Leftrightarrow$ $= [T]_\beta Q$

$\Leftrightarrow$ $[I_V \circ T]$ $[T \circ I_V]$

* 5 problems, last one is proof.

solved question

2.4

#9 #20

$I_V: V \to W$ (V: β; W: β')

$[I_V]_\beta^{\beta'} = Q$.

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

2.1 Linear Transformations, Null spaces, and Ranges

1 3 9 11 12 13 14 15 16 17

Example 10

Define the linear transformation $T: P_2(R) \to M_{2\times 2}(R)$ by

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Since $\beta = \{1, x, x^2\}$ is a basis for $P_2(R)$, we have

$$\begin{aligned} R(T) &= \text{span}(T(\beta)) = \text{span}(\{T(1), T(x), T(x^2)\}) \\ &= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \\ &= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right). \end{aligned}$$

Thus we have found a basis for $R(T)$, and so $\dim(R(T)) = 2$. ◆

$R(T) = \text{span}(T(\beta))$
$= \text{span}(\{T(1), T(x), T(x^2)\})$

Example 11 $\beta = \{1, x, x^2\}$

$T: P_2(R) \to P_3(R)$

$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$

$\int_0^x 3t^2\,dt$

$R(T) = \text{span}(\{T(\beta)\})$
$= \text{span}(\{T(1), T(x), T(x^2)\})$
$= \text{span}(\{3x, 2+\frac{3}{2}x^2, 4x+t^3\})$

linearly independent,

$\text{rank}(T) = 3$

$\dim(P_2(R)) = \text{rank}(T) + \text{nullity}(T)$

$3 = 3 + 0.$

T is one-to-one.

For Exercises 2 through 6, prove that T is a linear transformation and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

3. $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

① prove T is LT

$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

let $c \in R$, $x, y \in R^2$

where $x = (b_1, b_2)$
$y = (d_1, d_2)$

$cx + y = (cb_1 + d_1, cb_2 + d_2)$

$T(cx + y) = (cb_1 + d_1 + cb_2 + d_2, 0, 2cb_1 + d_1 - cb_2 - d_2)$

$= (c(b_1 + b_2) + d_1 + d_2, 0, c(2b_1 - b_2) + d_1 - d_2)$

$\Rightarrow T(cx) + T(y)$

$= cT(x) + T(y)$

so T is linear

② find bases for both N(T) and R(T)

$R(T) = \text{Span}\{T(\beta)\}$

$= \text{Span}\{T(a_1), T(a_2)\}$

$= \text{Span}\{(1, 0, 2), (1, 0, -1)\}$

They are linearly independent.

so $\dim(R(T)) = 2$

By dimension theorem,

$\dim(R^2) = \dim(R(T)) + \dim(N(T))$

$2 = 2 + 0$

$\dim(N(T)) = 0$

$\text{Nullity}(T) = \{0\}$.

③ T is onto? one-to-one?

first, dimension $\dim(R^2) \leq \dim(R^3)$

which is leading to one-to-one

Also, nullity(T) = {0}, it's one-to-one

9. In this exercise, $T: R^2 \to R^2$ is a function. For each of the following parts, state why T is *not* linear.

(a) $T(a_1, a_2) = (1, a_2)$
(b) $T(a_1, a_2) = (a_1, a_1^2)$
(c) $T(a_1, a_2) = (\sin a_1, 0)$
(d) $T(a_1, a_2) = (|a_1|, a_2)$
(e) $T(a_1, a_2) = (a_1 + 1, a_2)$

(c) $T(a_1, a_2) = (\sin a_1, 0)$

let $c \in R$ $x, y \in R^2$

$x = (x_1, x_2)$ $y = (y_1, y_2)$

$cx + y = (cx_1 + y_1, cx_2 + y_2)$

$T(cx + y) = (\sin(cx_1 + y_1), 0)$

$T(cx) = (\sin cx_1, 0)$ $\neq$
$T(y) = (\sin y_1, 0)$

$T(cx) + T(y) = (\sin cx_1 + \sin y_1, 0)$

$n \le m$
one-to-one

$n \ge m$ onto.

11. Prove that there exists a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

[1] note that $\{u_1 = (1,1), u_2 = (2,3)\}$ is LI
there fore a basis of $\mathbb{R}^2$.

[2] Any vector $x = (x_1, x_2)$
$= (x_1, x_2) = a(1,1) + b(2,3)$

$$\begin{aligned} a+2b &= x_1 \\ -)\ a+3b &= x_2 \\ \hline -b &= -x_2 + x_1 \\ b &= x_2 - x_1 \end{aligned}$$

$a + 2(x_2 - x_1) = x_1$
$a = 3x_1 - 2x_2$

$(x_1, x_2) = (3x_1 - 2x_2)(1,1) + (x_2 - x_1)(2,3)$

$$\begin{aligned} T(x_1, x_2) &= 3x_1 - 2x_2\, T(1,1) + (x_2 - x_1)\, T(2,3) \\ &= (3x_1 - 2x_2)(1,0,2) + (x_2 - x_1)(1,-1,4) \\ &= (3x_1 - 2x_2,\ 0,\ 6x_1 - 4x_2) \\ &\quad + (-x_1 + x_2,\ -x_2 + x_1,\ 4x_2 - 4x_1) \\ \hline &\quad (2x_1 - x_2,\ -x_2 + x_1,\ 2x_1) \end{aligned}$$

[3] $T(8,11) = (16-11,\ -11+8,\ 16)$
$= (5, -3, 16)$ ∎

12. Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

[1] note that $\{u_1 = (1,0,3), u_2 = (-2,0,6)\}$
is not linearly independence,
would not satisfy the def of LT

indeed,
$T(-2,0,-6) = T(-2(1,0,3)) = (2,1)$
$-2\,T((1,0,3)) = (-2,-2)$

$(2,1) \ne (-2,-2)$ ∎

13. Let V and W be vector spaces, let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $\mathsf{R}(\mathsf{T})$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathsf{T}(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

Sol)

Suppose $a_1, a_2, \ldots, a_k$ are arbitrary scalars
such that $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \cdot V$
we must show that $a_1 = a_2 = \cdots = a_k = 0$

we have $T(a_1v_1 + \cdots + a_kv_k) = T(0 \cdot V)$ which is
$0 \cdot W$ since T is linear.
Also, by linearity, $T(a_1v_1 + \cdots + a_kv_k)$
$= a_1T(v_1) + a_2T(v_2) + \cdots + a_kT(v_k)$
$= a_1w_1 + a_2w_2 + \cdots + a_kw_k = 0 \cdot W$ since
this equal to the $0 \cdot W$
and $w_1, w_2, \ldots, w_k$ is LI
so we conclude that $a_1, a_2, \ldots a_k = 0$, as required ∎

14. Let V and W be vector spaces and $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $\mathsf{T}(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $\mathsf{T}(\beta) = \{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}$ is a basis for W.

(a) The sufficiency is due to that if $T(x) = 0$, $\{x\}$ can not be independent and hence $x = 0$. For the necessity, we may assume $\sum a_iT(v_i) = 0$. Thus we have $T(\sum a_iv_i) = 0$. But since T is one-to-one we have $\sum a_iv_i = 0$ and hence $a_i = 0$ for all proper i.

(b) The sufficiency has been proven in Exercise 2.1.13. But note that S may be an infinite set. And the necessity has been proven in the previous exercise.

(c) Since T is one-to-one, we have $T(\beta)$ is linear independent by the previous exercise. And since T is onto, we have $R(T) = W$ and hence $\text{span}(T(\beta)) = R(T) = W$.

if T is onto $R(T) = W$

$R(T) = \text{Span}\{T(\beta)\} = W$

15. Recall the definition of $\mathsf{P}(R)$ on page 10. Define

$$\mathsf{T}\colon \mathsf{P}(R) \to \mathsf{P}(R) \quad \text{by} \quad \mathsf{T}(f(x)) = \int_0^x f(t)\,dt$$

Prove that ① T linear and ② one-to-one, but ③ not onto.

[1] To prove T is linear, we need to show that

$T(cf(x) + g(x)) = cT(f(x)) + T(g(x))$

$f, g \in P(R)$, scalar c.

$T(cf(x) + g(x)) = \int_0^x cf(t) + g(t)\,dt$

$= cf(x) + g(x)$

$= cT(f(x)) + T(g(x))$ □

$\therefore$ T is linear.

[2] To prove T is one-to-one we need to show that

$T(f(x)) = T(g(x))$ implies $f(x) = g(x)$

Suppose $T(f(x)) = T(g(x))$

$\int_0^x f(t)\,dt = \int_0^x g(t)\,dt$

$f(x) = g(x)$.

$\therefore$ So T is on-to-one

[3] To prove T is not onto, we need to show that there's a $g \in P(R)$ such that $T(f(x)) \neq g(x)$ for all $f \in P(R)$.

Let $g(x) = c$, where c is nonzero constant.

Suppose $T(f(x)) = g(x)$.

Then $\int_0^x f(t)\,dt = c$ $\qquad f(x) = 0$

$F(x) = C \qquad F'(x) = f(x)$

$F'(x) = 0 \qquad f(x) = 0$.

But if $f(x) = 0$, then $T(f(x)) = \int_0^x 0 \cdot dt = 0$ which is not c.

$\therefore$ So $T(f(x)) \neq g(x)$. Therefore T is not onto

16. Let $\mathsf{T}\colon \mathsf{P}(R) \to \mathsf{P}(R)$ be defined by $\mathsf{T}(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

[1] To prove T is onto, we need to show that for any element $g(x)$ in $P(R)$ we can find

$f(x) = \int g(x)\,dx$ in $P(R)$ which satisfies $T(f(x)) = g(x)$

↳ $g(x)$ is in $R(T)$

$P(R) \subseteq R(T)$

$R(T) \subseteq P(R)$

$R(T) = P(R) \rightarrow$ T is onto.

[2] To prove T is not one-to-one

$T(1) = T(2) = 0$

$N(T) \neq \{0\}$.

$\rightarrow$ T is not one-to-one

17. Let V and W be finite-dimensional vector spaces and $T: V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

(a) Suppose T is onto

Then by theorem, $\underbrace{\text{rank}(T)}_{\dim(R(T))} = \dim(W)$

By dimension theorem

$\dim(V) = \text{rank}(T) + \text{nullity}(T)$

$\text{rank}(T) \leq \dim(W)$..

By we assume

$\dim(W) > \text{rank}(T) + \text{nullity}(T)$

$\dim(W) > \dim(W) + \text{nullity}(T)$

$0 > \text{nullity}(T)$

contradiction.. nullity is always a nonnegative integer.

(b) Suppose T is one-to-one

Then by theorem $\dim(V) = \text{rank}(T)$

since $\text{nullity}(T) = 0$

By dimension theorem

$\dim(V) = \text{rank}(T) + \text{nullity}(T)$

$\dim(W) < \text{rank}(T)$

But this is clearly impossible bc

$R(T)$ is a subspace of W and

therefore always has dimension less than equal

to the dimension of W

2.2 The matrix representation of a linear transformation

1 2(a) 5(b,c)
10 16

gonna be 3×2 matrix

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.
   (a) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

Sol for (a)

$T(a_1) = (2, 3, 1)$

$T(a_2) = (-1, 4, 0)$

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$ ∎

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

gonna be 4×3 matrix

(b) Define

$$\mathsf{T}\colon \underbrace{\mathsf{P}_2(R)}_{\beta} \to \underbrace{\mathsf{M}_{2\times 2}(R)}_{\alpha} \quad \text{by} \quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$.

(c) Define $\mathsf{T}\colon \underbrace{\mathsf{M}_{2\times 2}(F)}_{\alpha} \to \underbrace{F}_{\gamma}$ by $\mathsf{T}(A) = \mathrm{tr}(A)$. Compute $[\mathsf{T}]_\alpha^\gamma$.

Sol for (b)

$T: P_2(R) \longrightarrow M_{2\times 2}(R)$
$\{1, x, x^2\}$

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Sol for (b)

$T: M_{2\times 2}(F) \longrightarrow F$ by $T(A) = tr(A)$
$\{E^{11}, E^{12}, E^{21}, E^{22}\}$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$$[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$ ∎

10. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6 (p. 72), there exists a linear transformation $T: V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[T]_\beta$.

$\beta = \{v_1, v_2, \ldots v_n\}$

Define $v_0 = 0$ ..

$T(v_1) = v_1 + v_0 = v_1$

$T(v_2) = v_2 + v_1$

$T(v_3) = v_3 + v_2$

$\vdots$

$T(v_n) = v_n + v_{n-1}$

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [T]_\beta$$

16. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T: V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

$\boxed{1}$ Let $n = \dim V = \dim W$
and $k = \text{nullity}(T)$

choose a basis $\beta = \{v_1, v_2 \ldots, v_k, v_{k+1} \ldots, v_n\}$ of V with $\{v_1 \ldots, v_k\}$ a basis of $N(T)$
(extend any basis of $N(T)$ to a basis of V.)

$\boxed{2}$ Set $u_j = T(v_j)$ for $j = k+1, \ldots, n$
Then $\{u_{k+1} \ldots, u_n\}$ is a basis of $R(T)$: it spans because T of a basis spans the range
It's independence since $\sum_{j=k+1}^{n} c_j u_j = 0 \Rightarrow T\left(\sum_{j=k+1}^{n} c_j v_j\right) = 0,$

So $\sum_{k+1}^{n} c_j v_j \in N(T) = \text{span}\{v_1 \ldots, v_k\}$, which forces all $c_j = 0$ by uniqueness of

coordinates in the basis β.
Because $\dim W = n$
$\dim R(T) = n-k$, extend to a basis of W:
$\gamma = \{u_1 \ldots, u_k, u_{k+1} \ldots, u_n\}$

$\boxed{3}$ Compute columns

For $j \le k$, $T(v_j) = 0 \to$ column j is 0
For $j \ge k+1$ $T(v_j) = u_j \to$ column j is the γ-coordinate vector e_j

Hence $[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k} \end{pmatrix} = \text{diag}(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{n-k})$ a diagonal matrix

2.3 Composition of linear Transformations and matrix multiplication

1 2(b) 3 4(a,b) 9 10 11 12 13

$T: \underset{\alpha}{V} \to \underset{\beta}{W}$ and $U: \underset{\beta}{W} \to \underset{\gamma}{Z}$

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

$T: \underset{\beta}{V} \to \underset{\gamma}{W}$ $u \in V$

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

$L_A : F^n \to F^m$

$L_A(x) = Ax$ for each column vector $x \in F^n$

L_A a left-multiplication transformation

Ex 4

$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}_{2\times 3}$

$A \in M_{2\times 3}(R)$ and $L_A : R^3 \to R^2$

if $x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$L_A(x) = Ax = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

* matrix multiplication is associative

$A(BC) = (AB)C$

* matrix multiplication is not commutative

$AB \neq BA$

2.3 Composition of Linear Transformation and Matrix Multiplication

1 2(b) 3 4(a,b) 9 11 12 13

T/F — calculation — proof

3. Let $g(x) = 3 + x$. Let $\underbrace{\mathsf{T}\colon \overset{\beta}{\mathsf{P}_2(R)} \to \overset{\beta}{\mathsf{P}_2(R)}}_{3\times3 \text{ matrix}}$ and $\underbrace{\mathsf{U}\colon \overset{\beta}{\mathsf{P}_2(R)} \to \overset{\gamma}{\mathsf{R}^3}}_{3\times3\text{ matrix}}$ be the linear transformations respectively defined by $\{1, x, x^2\}$

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \quad\text{and}\quad U(a + bx + cx^2) = (a + b, c, a - b).$$

$= f'(x)(3+x)+2f(x)$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(R)$ and R^3, respectively.

(a) Compute $[\mathsf{U}]_\beta^\gamma$, $[\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

Sol for (a)

$\boxed{1}$ compute $[U]_\beta^\gamma$

$U(1) = (1,0,1)$

$U(x) = (1,0,-1)$

$U(x^2) = (0,1,0)$

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\boxed{2}$ compute $[T]_\beta$

$T(1) = 0 + 2\cdot 1 = 2$

$T(x) = 1\cdot(3+x) + 2x$

$= 3x + 3$

$T(x^2) = 2x(3+x) + 2x^2$

$= 4x^2 + 6x$

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$\boxed{3}$ $[UT]_\beta^\gamma = [U]_\beta^\gamma\,[T]_\beta$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$$

Sol for (b)

$\boxed{1}$ Compute $[h(x)]_\beta$

$h(x) = 3 - 2x + x^2$

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\boxed{2}$ compute $[U(h(x))]_\gamma$

$[U]_\beta^\gamma \cdot [h(x)]_\beta$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.14 to compute the following vectors:

(a) $[\mathsf{T}(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

(b) $[\mathsf{T}(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

Define $\mathsf{T}\colon \mathsf{M}_{2\times2}(F) \to \mathsf{M}_{2\times2}(F)$ by $\mathsf{T}(A) = A^t$. Compute $[\mathsf{T}]_\alpha$.

Define $\{E^{11}, E^{12}, E^{21}, E^{22}\}$

$$\mathsf{T}\colon \underset{\beta}{\mathsf{P}_2(R)} \to \underset{\alpha}{\mathsf{M}_{2\times2}(R)} \quad\text{by}\quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

Let

$$\alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

$\beta = \{1, x, x^2\}$,

and

$\gamma = \{1\}$.

Sol for (a)

$[T(A)]_\alpha = [T]_\alpha^\alpha \cdot [A]_\alpha$

$= [T]_\alpha \cdot [A]_\alpha$

$\boxed{1}$ compute $[T]_\alpha$

$T(E^{11}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$T(E^{12}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$T(E^{21}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T(E^{22}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{2}$ compute $[A]_\alpha$

$$= \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix}$$

$\boxed{3}$ $[T(A)]_\alpha$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$$

Sol for (b)

$[T(f(x))]_\alpha = [T]_\beta^\alpha \cdot [f(x)]_\beta$

$\boxed{1}$ compute $[T]_\beta^\alpha$

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$8 - \overset{14}{12} + 6$

$\boxed{2}$ compute $[f(x)]_\beta$

$$[4 - 6x + 3x^2]_\beta = \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$$

$\boxed{3}$ $[T(f(x))]_\alpha$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$$

9. Find linear transformations $\mathsf{U}, \mathsf{T}: \mathsf{F}^2 \to \mathsf{F}^2$ such that $\mathsf{UT} = \mathsf{T}_0$ (the zero transformation) but $\mathsf{TU} \neq \mathsf{T}_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

Let $U(x,y) = (y,0)$

$T(x,y) = (x,0)$

$UT(x,y) = U(x,0) = (0,0) = T_0$

$TU(x,y) = T(y,0) = (y,0) \neq T_0$

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$

11. Let V be a vector space, and let $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ be linear. Prove that $\mathsf{T}^2 = \mathsf{T}_0$ if and only if $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$.

$\Rightarrow$ Suppose $T^2 = T_0$

for any vector $v \in V$

we have $T(T(v)) = T^2(v) = T_0(v) = 0$

Therefore, $T(v) \in N(T)$ for every $v \in V$

$\Leftarrow$ Suppose $R(T) \subseteq N(T)$

for any vector $v \in V$

we know $T(v) \in R(T)$, and $R(T) \subseteq N(T)$,

it follows $T(v) \subseteq N(T)$,

Then $T^2(v) = T(T(v)) = 0$ $\boxtimes$

12. Let V, W, and Z be vector spaces, and let T: V → W and U: W → Z be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?
(b) Prove that if UT is onto, then U is onto. Must T also be onto?
(c) Prove that if U and T are one-to-one and onto, then UT is also.

Sol for (a)
$\boxed{1}$ Show T is 1-1
suppose UT is 1-1
Take $v \in N(T)$
Then $UT(v) = U(T(v)) = U(0) = 0$
Since UT is 1-1, only vector mapping to is $v=0$
$N(T) = \{0\}$
$\therefore$ T is one-to-one
$\boxed{2}$ Must U also be 1-1?
counter example
Take $U: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $U(x,y,z) = (x,y)$
$T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x,y,0)$

Now the composition
$UT(x,y) = U(x,y,0) = (x,y)$
That's exactly the identity map on $\mathbb{R}^2$

There fore U may not be one-to-one

Sol for (b)
$\boxed{1}$ Show U is onto
Suppose UT is onto
take any $z \in Z$, $\exists v \in V$ st $UT(v) = z$
Since UT is onto, $\exists\, UT(v) = U(T(v))$
So every $z \in Z$ is the image of something under U.
That means U is onto

$\boxed{2}$ must T also be onto?
Take $U(x,y,z)$
same counter example

Sol for (c)
$\boxed{1}$ Show that UT is one-to-one
suppose $v \in N(T)$
Then $UT(v) = 0$ $U(T(v)) = 0$
Since U is one-to-one, this forces $T(v) = 0$
Since T is one-to-one, this forces $v = 0$
Then $N(UT) = \{0\}$
UT is 1-1

$\boxed{2}$ Show that UT is onto
Take any $z \in Z$
Since U is onto, there exists some $w \in W$ with $U(w) = z$
Since T is onto, there exists some $v \in V$ with $T(v) = w$
so $UT(v) = U(T(v)) = U(w) = z$
thus, UT is onto ☒

13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.

① $$\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} A_{ij} \sum_{j=1}^{n} B_{ji} = \sum_{j=1}^{n} B_{ji} \sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} (BA)_{jj} = \operatorname{tr}(BA)$$

② $$\operatorname{tr}(A^t) = \sum_{i=1}^{n} (A^t)_{ii} = \sum_{i=1}^{n} A_{ii}$$

1 2(e, f) 9 16 17 20 proof

4 5 6 7 18 19 calculation

2.4 invertiblility and Isomorphisms

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
(b) $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.
(c) $T: R^3 \to R^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.
(d) $T: P_3(R) \to P_2(R)$ defined by $T(p(x)) = p'(x)$.
(e) $T: M_{2\times 2}(R) \to P_2(R)$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.
(f) $T: M_{2\times 2}(R) \to M_{2\times 2}(R)$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Sol for (c)
according to the dimension of T,

$T(a_1) = (3\ 0\ 3)$
$T(a_2) = (0\ 1\ 4)$
$T(a_3) = (-2, 0, 0)$

$[T] = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$

$T: \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3a_1 - 2a_3 \\ a_2 \\ 3a_1 + 4a_2 \end{pmatrix}$

(e) $T: M_{2\times2}(R) \to P_2(R)$ $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$

dimension 4 dimension 3

basis: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(E_{11}) = 1 + 0 + 0$
$T(E_{12}) = 0 + 2x + 0$
$T(E_{21}) = 0 + 0 + x^2$
$T(E_{22}) = 0 + 0 + x^2$

$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

→ Three rows clearly independent
so rank (T) = 3

By theorem, rank-nullity theorem.
dim (domain) = rank (T) + nullity (T)
$4 = 3 + 1$

∴ so T is not injective, not invertible

T is not invertible as it is a mapping between spaces of different dimensions

(f) $T: M_{2\times2}(R) \to M_{2\times2}(R)$ $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$

$T(E_{11}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
$T(E_{12}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$T(E_{21}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$
$T(E_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$[T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

-1 $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ -1

$+1$ $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$

$(+1)$ $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$

dim(domain) = rank(T) + nullity(T)
$4 = 4 + 0$

nullity (T) = 0

T is invertible

4.† Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

$$
\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A(B(B^{-1}A^{-1})) \\
&= A((BB^{-1})A^{-1}) \\
&= A(IA^{-1}) \\
&= AA^{-1} \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
(B^{-1}A^{-1})AB &= B^{-1}(A^{-1}AB) \\
&= B^{-1}(IB) \\
&= B^{-1}B \\
&= I
\end{aligned}
$$

5.† Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

$$
\begin{aligned}
(A^{-1})^t A^t &= (A^{-1}A)^t \\
&= I^t \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
A^t (A^{-1})^t &= (AA^{-1})^t \\
&= I^t \\
&= I
\end{aligned}
$$

6. Prove that if A is invertible and $AB = O$, then $B = O$.

If A is invertible, then A^{-1} exists

so we have $B = A^{-1}AB = A^{-1}O = O$

7. Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that A is not invertible.

(b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

Sol for (a)

$\boxed{\text{I}}$ Assume that A is invertible

Then
$$
\begin{aligned}
A^{-1}A^2 &= A^{-1}O \\
A^{-1}AA &= O \\
IA &= O \\
A &= O
\end{aligned}
$$

But this is a contradiction,

Since the zero matrix is not invertible

Sol for (b)

$\boxed{\text{I}}$ Assume that A is invertible and $b \neq O$

Then
$$
\begin{aligned}
A^{-1}AB &= A^{-1}O \\
IB &= O \\
B &= O
\end{aligned}
$$

Contradiction, Since $b \neq O$ by hypothesis

$\rightarrow$ $AB = O$ implies A not invertible ... ??

??? Square matrices

9. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are ~~invertible. Give~~ an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

Sol for 9.

If AB is invertible then L_{AB} is invertible so $L_A L_B = L_{AB}$ is surjective and injective

And thus L_A is injective and L_B surjective

But since their domain and codomain has the same dimension, they are both invertible

prof. answer.

$A_{n\times n}$, $B_{n\times n}$

AB is invertible $\Rightarrow$ A and B are both invertible

<Direct proof>

pf method 1: Let $C = (AB)^{-1}$ so Then AB must be exist??

$A(BC) = (AB)C = I$

because we assume $C = (AB^{-1})$

$\Rightarrow$ BC is inverse of A

$A \cdot B \quad B^{-1}A^{-1}$

idea of BC.

$(CA)B = C(AB) = I \rightarrow CA$ is inverse of B

pf method2: AB is invertible

$\Rightarrow L_{AB}$ is invertible

$\Rightarrow L_A \circ L_B$ is invertible

#2-3 Q.12

$\Rightarrow L_B$ is 1-1

$L_B : R^n \rightarrow R^n$

1-1 implies onto

Same dimension

$\Rightarrow L_B$ is onto

$\Rightarrow L_B$ is invertible

$\Rightarrow B$ is invertible

It is onto $\Rightarrow L_A$ is onto

$\Rightarrow L_A$ invertible

$\Rightarrow A$ is invertible

16. Let B be an $n \times n$ invertible matrix. Define $\Phi: \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

we can check Φ is linear since

$\boxed{1}$ $\Phi(A+cD) = B^{-1}(A+cD)B$

$= B^{-1}(AB) + B^{-1}cDB$

$= B^{-1}AB + cB^{-1}DB$

$= \Phi(A) + c\Phi(D)$

$\boxed{2}$ It's injective since if $\Phi(A) = B^{-1}AB = 0$

then we have $BB^{-1}ABB^{-1} = BOB^{-1}$

$A = BOB^{-1} = O$

$\boxed{3}$ It's surjective since for each D we have

that $\Phi(BDB^{-1}) = B^{-1}(BDB^{-1})B$

$= D.$

17.[†] Let V and W be finite-dimensional vector spaces and $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.
(b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

sol for (a)

If $y_1, y_2 \in T(V_0)$ and $y_1 = T(x_1)$, $y_2 = T(x_2)$,

we have $y_1 + y_2 = T(x_1 + x_2) \in T(V_0)$

$Cy = T(Cx_1) = T(V_0)$ $=?$ $(\in)$

since V_0 is a subspace so $0 = T(0) \in T(V_0)$

$T(V_0)$ is a subspace of W.

sol for (b)

we can cosider a mapping T' from

$\boxed{V_0 \text{ to } T(V_0)}$ by $T'(x) = T(x)$ for all $x \in V_0$

It's natural that T' is surjective

It's also injective since T is injective

so by Dimension Theorem,

$\dim(V_0) = \dim(N(T')) + \dim(R(T')) = \dim(T(V_0))$

$T': V_0 \to T(V_0)$

$T'(x) = T(x)$

$\dim(V_0) = \dim(N(T')) + \dim(R(T'))$

$= \dim(T(V_0))$

Example 7

Recall the linear transformation $\mathsf{T}: \mathsf{P}_3(R) \to \mathsf{P}_2(R)$ defined in Example 4 of Section 2.2 ($\mathsf{T}(f(x)) = f'(x)$). Let β and γ be the standard ordered bases for $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$, respectively, and let $\phi_\beta: \mathsf{P}_3(R) \to \mathsf{R}^4$ and $\phi_\gamma: \mathsf{P}_2(R) \to \mathsf{R}^3$ be the corresponding standard representations of $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$. If $A = [\mathsf{T}]_\beta^\gamma$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Consider the polynomial $p(x) = 2+x-3x^2+5x^3$. We show that $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma\mathsf{T}(p(x))$. Now

$$\mathsf{L}_A\phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

But since $\mathsf{T}(p(x)) = p'(x) = 1 - 6x + 15x^2$, we have

$$\phi_\gamma\mathsf{T}(p(x)) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

So $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma\mathsf{T}(p(x))$. ◆

Try repeating Example 7 with different polynomials $p(x)$.

18. Repeat Example 7 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.

Sol for 18. To show $\underbrace{L_A\phi_\beta(p(x))}_{[p(x)]_\beta} = \underbrace{\phi_\gamma T(p(x))}_{[T(p(x))]_\gamma}$

$\boxed{1}$ find L_A

$A = [T]_\beta^\gamma$

$T: P_3(R) \to P_2(R) \quad T(f(x)) = f'(x)$

$\beta = \{1, x, x^2, x^3\} \quad \gamma = \{1, x, x^2\}$

$T(1) = 0$
$T(x) = 1$
$T(x^2) = 2x$
$T(x^3) = 3x^2$

$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$

So $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$

$\boxed{2}$ find $\phi_\beta(p(x))$

$= [p(x)]_\beta$

$p(x) = 1 + x + 2x^2 + x^3$

$\phi_\beta(p(x)) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}_{4\times 1}$

$\boxed{3}$ find $\phi_\gamma T(p(x))$

① $T(p(x)) = T(1 + x + 2x^2 + x^3)$
$= 1 + 4x + 3x^2$

$\phi_\gamma T(p(x)) = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$

$\boxed{4}$ show $L_A\phi_\beta(p(x)) = \phi_\gamma T(p(x))$

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \phi_\gamma T(p(x))$ ∎

$L_A\phi_\beta(p(x)) = \phi_\gamma T(p(x))$

19. In Example 5 of Section 2.1, the mapping $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}(M) = M^t$ for each $M \in \mathsf{M}_{2\times 2}(R)$ is a linear transformation. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, which is a basis for $\mathsf{M}_{2\times 2}(R)$, as noted in Example 3 of Section 1.6.

(a) *Compute* $[\mathsf{T}]_\beta$.

(b) Verify that $\mathsf{L}_A\phi_\beta(M) = \phi_\beta\mathsf{T}(M)$ for $A = [\mathsf{T}]_\beta$ and

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Sol for 19. (a)

$\boxed{1}$ Compute $[T]_\beta$

$T: M_{2\times 2}(R) \to M_{2\times 2}(R) \quad T(M) = M^t$

$\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$

$T(E^{11}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad [T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$T(E^{12}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$T(E^{21}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T(E^{22}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$[M]_\beta \quad [T(M)]_\beta$

Sol for 19 (b)

verify that $L_A \phi_\beta(M) = \phi_\beta T(M)$ for $A = [T]_\beta$

$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\boxed{2}$ $[M]_\beta = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= 1E^{11} + 2E^{12} + 3E^{21} + 4E^{22}$

$\phi_\beta T(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$\boxed{3}$ $[T(M)]_\beta$

$T(M) = M^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_\beta = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$

$\boxed{4}$ $L_A \phi_\beta(M) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$

$\therefore L_A \phi_\beta(M) = \phi_\beta T(M)$ ∎

20.[†] Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{L}_A)$ and that $\text{nullity}(\mathsf{T}) = \text{nullity}(\mathsf{L}_A)$, where $A = [\mathsf{T}]_\beta^\gamma$. *Hint:* Apply Exercise 17 to Figure 2.2.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ R^n \arrow[r, "L_A"] & R^m \end{tikzcd}$$

β, n $\quad$ γ, m

$T(V)$

$R(T) \subseteq W$

$A_{m\times n} = [T]_\beta^\gamma$

$\phi_\gamma T(V) \subseteq R^m$

How can I understand this figure / diagram?

$L_A \phi_\beta = \phi_\gamma T$

$L_A \phi_\beta(V) = \phi_\gamma T(V)$

prove ① $\text{rank}(T) = \text{rank}(L_A)$ ② $\text{nullity}(T) = \text{nullity}(L_A)$

① : $\text{rank}(T) = \dim(R(T))$

$\text{rank}(L_A) = \dim(R(L_A))$

#17 and ϕ_γ is invertible

$R(T) = T(V)$

range of T

$*$ key step

$\dim(R(T)) = \dim(T(V)) = \dim(\phi_\gamma \circ T(V))$

SAME!!

$\phi_\gamma \circ T(V) = L_A \circ \phi_\beta(V)$

bc ϕ_β is onto

$\Rightarrow \dim(\phi_\gamma \circ T(V)) = \dim(L_A \circ \phi_\beta(V)) = \dim(L_A(R^n))$

doesn't need to be whole W

bc different dimension

so $T(V)$

② trivial from dimension theorem.

we already proved $\text{rank}(T) = \text{rank}(L_A)$ and $\dim(V) = n$.

By dimension theorem,

$\text{nullity}(T) = n - \text{rank}(T)$

$= n - \text{rank}(L_A)$

$= \text{nullity}(L_A)$

$\dim(V) = \text{rank}(T) + \text{nullity}(T)$

$\underbrace{\dim(F^n)}_{n} = \text{rank}(L_A) + \text{nullity}(L_A)$

$n = \text{rank}(T) + \text{nullity}(L_A)$

$\text{nullity}(L_A) = n - \text{rank}(T)$

$*$ $L_A : F^n \to F^m$

$L_A(x) = Ax$ $\quad$ for $m\times n$ matrix A.

- Rank of L_A

$\text{rank}(L_A) = \dim(\text{range}(L_A)) = \dim(\text{column space of } A)$

- nullity of L_A

$\text{nullity}(L_A) = \dim(N(L_A)) = \dim\{x \in F^n : Ax = 0\}$.

1. 2(a,c) 3(a)

5 7(a) 10

2.5 The change of coordinate matrix

2. For each of the following pairs of ordered bases β and β' for R^2, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$
(b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$
(c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$
(d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

(a) $\beta = \{\underset{(1,0)}{e_1}, \underset{(0,1)}{e_2}\}$ $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

$[(a_1, a_2)]_\beta = x(1,0) + y(0,1) = (a_1, a_2)$

$x = a_1 \quad y = a_2$

$[(b_1, b_2)]_\beta = x(1,0) + y(0,1) = (b_1, b_2)$

$x = b_1 \quad y = b_2$

$$Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$\downarrow$

$P: \beta \leftarrow \beta'$

(c) $\beta = \{(2,5), (-1,-3)\}$ $\beta' = \{\overset{(1,0)}{e_1}, \overset{(0,1)}{e_2}\}$

$[(1,0)]_\beta = x(2,5) + y(-1,-3) = (1,0)$

$$\begin{aligned} 2x - y &= 1 \\ 5x - 3y &= 0 \end{aligned}$$

$$\begin{aligned} 6x - \cancel{3y} &= 3 \\ -5x + \cancel{3y} &= 0 \end{aligned}$$

$x = 3$

$6 - y = 1$
$-y = -5$
$y = 5$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$[(0,1)]_\beta = x(2,5) + y(-1,-3) = (0,1)$

$$\begin{aligned} 2x - y &= 0 \\ 5x - 3y &= 1 \end{aligned}$$

$$\begin{aligned} 6x - \cancel{3y} &= 0 \\ -5x + \cancel{3y} &= -1 \end{aligned}$$

$x = -1$

$-2 - y = 0$
$-y = 2$
$y = -2$

$$= \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$$

3. For each of the following pairs of ordered bases β and β' for $P_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{x^2, x, 1\}$ and
$\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

Solution

$$[a_2x^2 + a_1x + a_0]_\beta = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix}$$

$$[b_2x^2 + b_1x + b_0]_\beta = \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}$$

$$[c_2x^2 + c_1x + c_0]_\beta = \begin{pmatrix} c_2 \\ c_1 \\ c_0 \end{pmatrix}$$

$$Q = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}$$

$[T]_\beta = Q[T]_{\beta'}Q^{-1}$

$\beta = \{1, x\}$.

5. Let T be the linear operator on $\mathsf{P}_1(R)$ defined by $\mathsf{T}(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1+x, 1-x\}$. Use Theorem 2.23 and the fact that

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[\mathsf{T}]_{\beta'}$.

$\boxed{1}$ $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T(1) = 0$
$T(x) = 1$

$\boxed{2}$ $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$P_{2\,\beta \leftarrow \beta'}$]

$[1+x]_\beta = a + bx = 1 + x \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$a = 1 \quad b = 1$

$[1-x]_\beta = a + bx = 1 - x \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$a = 1 \quad b = -1$

$\boxed{3}$ $Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -1 - 1 = -2$

$$Q^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$\boxed{4}$ $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \blacksquare$$

$*\ [T]_\beta = Q[T]_{\beta'}Q^{-1}$

$*\ [T]_{\beta'} = Q^{-1}[T]_\beta Q$

10. Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$. *Hint:* Use Exercise 13 of Section 2.3.

If A and B are similar,

we have $A = Q^{-1}BQ$ for some

invertible matrix Q

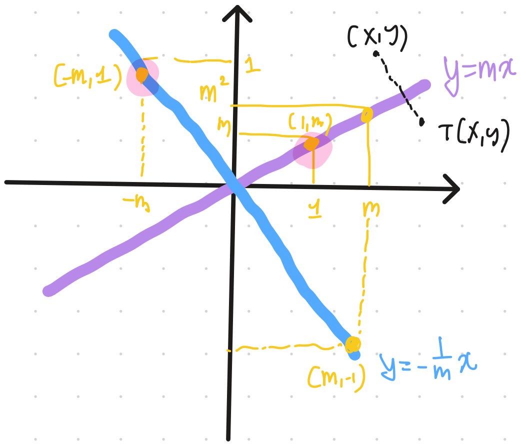
So we have

$$\mathrm{tr}(A) = \mathrm{tr}(Q^{-1}BQ)$$
$$= \mathrm{tr}(Q^{-1}QB)$$
$$= \mathrm{tr}(B) \blacksquare$$

$\{e_1, e_2\}$

7. In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $\mathsf{T}(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.



$$T(x,y) = [T]_\beta \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left(\frac{x(1-m^2)+2my}{1+m^2}, \frac{2mx + y(m^2-1)}{1+m^2} \right)$$

$$[T]_\beta = Q[T]_{\beta'}Q^{-1}$$

$$[T]_\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} [T]_{\beta'} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2}$$

$$= \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2}$$

$$= \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix} \frac{1}{1+m^2}$$

$\boxed{1}$ set β and β'

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \beta' = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix} \right\}$$

$\boxed{2}$ set $Q\ (\beta \leftarrow \beta')$

① $$\begin{bmatrix} 1 \\ m \end{bmatrix}_\beta = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$$
$$= \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$$

② $$\begin{bmatrix} -m \\ 1 \end{bmatrix}_\beta = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -m \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \qquad \det = 1+m^2$$

$\boxed{3}$ set Q^{-1}

$$Q^{-1} = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

$\boxed{4}$ set $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

① $$T\begin{pmatrix} 1 \\ m \end{pmatrix} = a\begin{pmatrix} 1 \\ m \end{pmatrix} + b\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} a-bm \\ am+b \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix}$$

$a - bm = 1$
$am + b = m$

$\cancel{am} - bm^2 = \cancel{m}$
$-\cancel{am} - b = -\cancel{m}$
$-b(m^2+1) = 0$

$a = 1$
$b = 0$

② $$T\begin{pmatrix} -m \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ m \end{pmatrix} + b\begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} m \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a-bm \\ am+b \end{pmatrix} = \begin{pmatrix} m \\ -1 \end{pmatrix}$$

$\cancel{am} - bm^2 = m^2$
$-\cancel{am} - b = 1$
$-b(m^2+1) = m^2+1$
$-b = 1$
$b = -1$

$a + m = m$
$a = 0$