MAS 4105 — Linear Algebra

University of Florida

Chapter 2 Study Guide

Linear Transformations & Matrix Representations

Textbook: Linear Algebra, 4th Edition

Friedberg, Insel & Spence

Self-teaching reference — MAS 4105, Dr. Rong

Contents

§1 Linear Transformations, Null Spaces, and Ranges

1.1 Definition and Basic Properties

Definition: Linear Transformation

Let V and W be vector spaces over a field F. A function $T : V \to W$ is a **linear transformation** if for all $x, y \in V$ and $c \in F$:

$$T(cx + y) = cT(x) + T(y)$$

Equivalently, T must satisfy both:
(i) **Additivity:** $T(x + y) = T(x) + T(y)$
(ii) **Homogeneity:** $T(cx) = cT(x)$

Note: To prove T is *not* linear, find a single counterexample. Common non-linear maps: $T(a_1, a_2) = (a_1, a_1^2)$, $T(a_1, a_2) = (|a_1|, a_2)$, $T(a_1, a_2) = (a_1 + 1, a_2)$, $T(a_1, a_2) = (\sin a_1, 0)$.

Theorem: Consequences of Linearity

If $T : V \to W$ is linear, then:
(a) $T(0_V) = 0_W$
(b) $T(-x) = -T(x)$ for all $x \in V$
(c) $T(x - y) = T(x) - T(y)$ for all $x, y \in V$
(d) $T\big(\sum_{i=1}^n a_i x_i\big) = \sum_{i=1}^n a_i T(x_i)$

1.2 Null Space and Range

Definition: Null Space and Range

Let $T : V \to W$ be linear. Define:

$$N(T) = \{x \in V : T(x) = 0_W\} \subseteq V \quad \text{(kernel)}$$

$$R(T) = \{T(x) : x \in V\} = T(V) \subseteq W \quad \text{(image)}$$

Both $N(T)$ and $R(T)$ are subspaces of their respective spaces.

Theorem: Range via a Basis

If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V, then

$$R(T) = \operatorname{span}\big(\{T(v_1), T(v_2), \ldots, T(v_n)\}\big)$$

Example: Computing $N(T)$ and $R(T)$

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 + a_2,\ 0,\ 2a_1 - a_2)$.

Range. Using $\beta = \{(1, 0), (0, 1)\}$:

$$T(1, 0) = (1, 0, 2), \quad T(0, 1) = (1, 0, -1).$$

These are linearly independent, so $R(T) = \text{span}\{(1,0,2),(1,0,-1)\}$ and $\text{rank}(T) = 2$.

Null space. Solve $T(a_1, a_2) = 0$: $a_1 + a_2 = 0$ and $2a_1 - a_2 = 0$ give $a_1 = a_2 = 0$. So $N(T) = \{(0,0)\}$, $\text{nullity}(T) = 0$, and T is one-to-one.

1.3 Dimension Theorem (Rank-Nullity)

Theorem: Dimension Theorem

Let $T : V \to W$ be linear with V finite-dimensional. Then:

$$\dim(V) = \text{rank}(T) + \text{nullity}(T)$$

Theorem: Injectivity and Surjectivity Criteria

Let $T : V \to W$ be linear.

(a) T is one-to-one $\iff$ $N(T) = \{0\}$
(b) If $\dim V = \dim W$: one-to-one $\iff$ onto
(c) $\dim V < \dim W \Rightarrow T$ cannot be onto
(d) $\dim V > \dim W \Rightarrow T$ cannot be one-to-one

Example: Existence of a Linear Transformation

Problem (§2.1 #11). Prove $\exists\, T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. Find $T(8,11)$.

Since $\{(1,1),(2,3)\}$ is linearly independent, it is a basis of $\mathbb{R}^2$, so T is uniquely and well-defined by specifying its values on these vectors. Write $(x_1, x_2) = a(1,1) + b(2,3)$:

$$a + 2b = x_1, \quad a + 3b = x_2 \implies a = 3x_1 - 2x_2,\ b = x_2 - x_1.$$

$$T(x_1, x_2) = (3x_1 - 2x_2)(1,0,2) + (x_2 - x_1)(1,-1,4) = (2x_1 - x_2,\ -x_1 + x_2,\ 2x_1).$$

So $T(8,11) = (5,-3,16)$.

Common Mistake: To show a linear transformation *does not exist*: check whether the given output vectors are consistent with linearity. If the input vectors are linearly dependent (e.g., $v_2 = 2v_1$) but the images do not satisfy $w_2 = 2w_1$, no such T exists.

Example: Proof: $T^2 = T_0 \iff R(T) \subseteq N(T)$

(⇒) Suppose $T^2 = T_0$. For any $v \in V$, $T(T(v)) = T^2(v) = 0$, so $T(v) \in N(T)$. Since v is arbitrary, $R(T) \subseteq N(T)$.

(⇐) Suppose $R(T) \subseteq N(T)$. For any $v \in V$, $T(v) \in R(T) \subseteq N(T)$, so $T(T(v)) = 0$. Hence $T^2(v) = 0$ for all v, i.e., $T^2 = T_0$.

---

§2 The Matrix Representation of a Linear Transformation

2.1 Coordinate Vectors

Definition: Coordinate Vector

Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for V. For any $v = \sum_{i=1}^n a_i v_i$, define:

$$[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

2.2 Matrix Representation

Definition: Matrix Representation $[T]_\beta^\gamma$

Let V, W have ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$. The **matrix representation** of $T : V \to W$ is the $m \times n$ matrix whose j-th column is $[T(v_j)]_\gamma$:

$$[T]_\beta^\gamma = \Big([T(v_1)]_\gamma \;\Big|\; [T(v_2)]_\gamma \;\Big|\; \cdots \;\Big|\; [T(v_n)]_\gamma \Big).$$

Note: **Dimensions:** $[T]_\beta^\gamma$ is $m \times n$ where $m = \dim W$ and $n = \dim V$. For example, $T : \mathbb{R}^2 \to \mathbb{R}^3$ gives a 3×2 matrix.

Theorem: Key Computational Theorem (Thm. 2.14)

For any $u \in V$:

$$[T(u)]_\gamma = [T]_\beta^\gamma \cdot [u]_\beta.$$

Coordinate vector of output = matrix × coordinate vector of input.

Example: Computing $[T]_\beta^\gamma$

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (2a_1 - a_2,\ 3a_1 + 4a_2,\ a_1)$, standard bases.

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

(b) $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$, $\beta = \{1, x, x^2\}$, α standard basis of $M_{2\times 2}$.

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) $T : M_{2\times 2}(F) \to F$, $T(A) = \mathrm{tr}(A)$, $\alpha = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, $\gamma = \{1\}$.

$$[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

§3 Composition and Matrix Multiplication

Theorem: Composition $\leftrightarrow$ Matrix Multiplication

Let $T : V \to W$ and $U : W \to Z$ be linear, with ordered bases α, β, γ for V, W, Z. Then:

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta.$$

Also, for any $v \in V$: $[T(v)]_\beta = [T]_\alpha^\beta \cdot [v]_\alpha$.

Definition: Left-Multiplication Transformation

For $A \in M_{m\times n}(F)$, define $L_A : F^n \to F^m$ by $L_A(x) = Ax$. Then:

- $[L_A]_\beta^\gamma = A$ (standard bases).
- $L_A = L_B \iff A = B$.
- $L_{A+B} = L_A + L_B$ and $L_{AB} = L_A \circ L_B$.

Theorem: Properties of Matrix Multiplication

(a) **Associativity:** $(AB)C = A(BC)$.
(b) **Distributivity:** $A(B + C) = AB + AC$.
(c) **Identity:** $I_m A = A = AI_n$.
(d) **Not commutative:** $AB \neq BA$ in general.
(e) **No cancellation:** $AB = AC \not\Rightarrow B = C$; $AB = O \not\Rightarrow A = O$ or $B = O$.

Example: $AB = O$ but $BA \neq O$

Let $U(x, y) = (y, 0)$ and $T(x, y) = (x, 0)$ on F^2. Then $UT = T_0$ but $TU \neq T_0$.
With $A = [U]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O, \qquad BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq O.$$

Theorem: Trace Properties

For $n \times n$ matrices A, B: $\quad \mathrm{tr}(AB) = \mathrm{tr}(BA) \quad$ and $\quad \mathrm{tr}(A^t) = \mathrm{tr}(A)$.

Proof of $\mathrm{tr}(AB) = \mathrm{tr}(BA)$*:*

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_i \sum_k A_{ik} B_{ki} = \sum_k \sum_i B_{ki} A_{ik} = \mathrm{tr}(BA).$$

§4 Invertibility and Isomorphisms

Definition: Invertible Transformation

$T : V \to W$ is **invertible** if $\exists$ linear $U : W \to V$ such that $UT = I_V$ and $TU = I_W$. Such U is unique and denoted T^{-1}.

Theorem: Invertibility $\iff$ Bijection

$T : V \to W$ is invertible $\iff$ T is both one-to-one and onto.

Theorem: Invertibility Implies Equal Dimensions

If $T : V \to W$ is invertible, then $\dim V = \dim W$.

Proof. T one-to-one $\Rightarrow$ nullity$(T) = 0$; T onto $\Rightarrow R(T) = W$, so rank$(T) = \dim W$. By the dimension theorem: $\dim V = 0 + \dim W$.

Theorem: Matrix of the Inverse

If $T : V \to W$ is invertible, then $[T]_\beta^\gamma$ is invertible and

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}.$$

4.1 The 2×2 Inverse Formula

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^{-1} = \frac{1}{xw - yz} \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}.$$

The scalar $xw - yz$ is the **determinant**. The matrix is invertible $\iff$ det $\neq 0$.

Example: Finding A^{-1}

Let $A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$. Then $\det(A) = 15 - 14 = 1$, so

$$A^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}.$$

Theorem: Properties of Matrix Inverses

Let A, B be invertible $n \times n$ matrices.

(a) $(AB)^{-1} = B^{-1}A^{-1}$

(b) $(A^{-1})^{-1} = A$

(c) $(A^t)^{-1} = (A^{-1})^t$

(d) If $AB = O$ and A is invertible, then $B = O$.

Example: Composition Invertibility (§2.4 #12)

Let $T : V \to W$ and $U : W \to Z$ be linear.

(a) If UT is 1-1, then T is 1-1. Let $v \in N(T)$. Then $T(v) = 0$, so $UT(v) = 0$. Since UT is 1-1, $v = 0$.

(b) If UT is onto, then U is onto. For any $z \in Z$, since UT is onto, $\exists\, v$ with $z = UT(v) = U(T(v)) \in R(U)$.

(c) If U and T are both 1-1 and onto, so is UT. For 1-1: if $v \in N(UT)$, then $U(T(v)) = 0$; since U is 1-1, $T(v) = 0$; since T is 1-1, $v = 0$. For onto: given z, since U is onto $\exists\, w$ with $U(w) = z$; since T is onto $\exists\, v$ with $T(v) = w$; so $UT(v) = z$.

Definition: Isomorphism

An invertible linear transformation $T : V \to W$ is an **isomorphism**. We say $V \cong W$ (*isomorphic*) if such a T exists.

Two finite-dimensional vector spaces over the same field are isomorphic $\iff$ they have the same dimension.

§5 The Change of Coordinate Matrix

Definition: Change of Coordinate Matrix

Let β and β' be two ordered bases for V. The **change of coordinate matrix from β' to β** is

$$Q = [I_V]_{\beta'}^{\beta}.$$

Its j-th column is $[v_j']_\beta$ where v_j' is the j-th vector of β'. For any $v \in V$:

$$[v]_\beta = Q\,[v]_{\beta'}.$$

Q is always invertible; Q^{-1} converts β-coordinates back to β'-coordinates.

Theorem: Changing Basis for a Linear Operator

Let $T : V \to V$ be linear and $Q = [I_V]_{\beta'}^{\beta}$. Then:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \qquad \Longleftrightarrow \qquad [T]_\beta = Q\,[T]_{\beta'}\,Q^{-1}.$$

Example: Computing the Change of Coordinate Matrix (§2.5 #5)

Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be $T(p(x)) = p'(x)$, $\beta = \{1, x\}$, $\beta' = \{1 + x,\ 1 - x\}$.

Step 1: $[T]_\beta$**.** $T(1) = 0$ and $T(x) = 1$, so $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Step 2: Q**.** Express β'-vectors in terms of β: $[1 + x]_\beta = \binom{1}{1}$ and $[1 - x]_\beta = \binom{1}{-1}$, so $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Step 3: Q^{-1}**.** $\det(Q) = -2$, so $Q^{-1} = \dfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\dagger} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$.

Step 4: $[T]_{\beta'} = Q^{-1}[T]_\beta Q$**.**

$$[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Example: Reflection about $y = mx$ (§2.5 #7)

The reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$ about $y = mx$.

Choose $\beta' = \left\{ \binom{1}{m}, \binom{-m}{1} \right\}$ (along and perpendicular to the line). In this basis:

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With $Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$ and $Q^{-1} = \dfrac{1}{1 + m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$:

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{1 + m^2}\begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix}.$$

So $T(x,y) = \dfrac{1}{1+m^2}\big((1-m^2)x + 2my,\ 2mx + (m^2-1)y\big)$.

Definition: Similar Matrices

$A, B \in M_{n\times n}(F)$ are **similar** if $\exists$ invertible Q such that $B = Q^{-1}AQ$.

$[T]_\beta$ and $[T]_{\beta'}$ are always similar. If $A \sim B$, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

§6 Exam 2 Preparation Summary

6.1 Master Theorem Checklist

§	**Key Facts**
2.1	T linear: $T(cx+y) = cT(x) + T(y)$. $\dim V = \text{rank}(T) + \text{nullity}(T)$. T 1-1 $\iff$ $N(T) = \{0\}$. $\dim V = \dim W$: 1-1 $\iff$ onto.
2.2	$[T]_\beta^\gamma$ is $m \times n$; j-th col $= [T(v_j)]_\gamma$. $[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.
2.3	$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$. $(AB)_{ij} = \sum_k A_{ik}B_{kj}$. $\text{tr}(AB) = \text{tr}(BA)$.
2.4	T invertible $\iff$ 1-1 and onto. $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. $(AB)^{-1} = B^{-1}A^{-1}$.
2.5	$Q = [I_V]_{\beta'}^\beta$: converts $\beta' \to \beta$ coords. $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. $[T]_\beta$ and $[T]_{\beta'}$ are similar.

6.2 Frequently Tested Proof Techniques

Proving T is linear: Show $T(cx+y) = cT(x) + T(y)$ for arbitrary $c \in F$ and $x, y \in V$.

Proving T is 1-1: Show $N(T) = \{0\}$, i.e., $T(v) = 0 \Rightarrow v = 0$.

Proving T is onto: Given arbitrary $w \in W$, construct $v \in V$ with $T(v) = w$.

Proving $T(\beta)$ is a basis of W: If T is 1-1 and onto and β is a basis of V, then $T(\beta)$ spans W (since T is onto) and is linearly independent (since T is 1-1).

Proving $T^2 = T_0 \iff R(T) \subseteq N(T)$: See §2.1 example above.

AB invertible $\Rightarrow$ A and B invertible ($n \times n$): AB invertible $\Rightarrow L_{AB} = L_A \circ L_B$ invertible. By §2.3 #12, L_B is 1-1 and L_A is onto. Since $L_A, L_B : F^n \to F^n$ (same dimension), each is invertible.

6.3 Step-by-Step: Computing $[T]_{\beta'}$

Given a linear operator $T : V \to V$ and two bases β, β':

1. Compute $[T]_\beta$: apply T to each β-vector, express in β-coords, form as columns.
2. Build $Q = [I_V]_{\beta'}^\beta$: express each β'-vector in β-coords, form as columns.
3. Compute Q^{-1} via the 2×2 formula or row reduction.
4. Multiply: $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

Check: $[T]_\beta = Q[T]_{\beta'}Q^{-1}$ should also hold.

6.4 Problem Index by Section

Section	Problem Type	Numbers
§2.1	Prove T linear; find $N(T), R(T)$; Dim Thm	3, 9, 11–17
§2.2	Compute $[T]_\beta^\gamma$; use Thm 2.14	2(a), 5(b)(c), 10, 16
§2.3	Compute $[UT]$; matrix products; trace	2(b), 3, 4(a)(b), 9, 11–13
§2.4	Invertibility; compute T^{-1}; proofs	2, 4, 5, 7, 9, 16–20
§2.5	Compute Q; change of basis; reflection	2, 3(a), 5, 7(a), 10

6.5 Diagonal Matrix via Change of Basis (§2.2 #16)

Claim: If $\dim V = \dim W$ and $T : V \to W$, $\exists$ bases β, γ such that $[T]_\beta^\gamma$ is diagonal.

Construction: Let $k = \text{nullity}(T)$. Extend a basis $\{v_1, \ldots, v_k\}$ of $N(T)$ to a basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Set $u_j = T(v_{k+j})$ for $j = 1, \ldots, n-k$; extend $\{u_1, \ldots, u_{n-k}\}$ to a basis γ of W. Then:

$$[T]_\beta^\gamma = \begin{pmatrix} 0_k & 0 \\ 0 & I_{n-k} \end{pmatrix} = \operatorname{diag}(\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{n-k}).$$